

Period doubling route to chaos in Taylor-Green dynamo

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Abstract. - We perform spectral simulations of dynamo for magnetic Prandtl number of one with Taylor-Green forcing. We observe dynamo transition through a supercritical pitchfork bifurcation. Beyond the transition, the numerical simulations reveal complex dynamo states with windows of constant, periodic, quasiperiodic, and chaotic magnetic field configurations. For some forcing amplitudes, multiple attractors were obtained for different initial conditions. We show that one of the chaotic windows follows the period-doubling route to chaos.

The phenomenon of spontaneous generation of magnetic fields in magnetohydrodynamics (MHD) is known as the dynamo effect. This is believed to be the generating mechanism for the magnetic field in astrophysical bodies such as planets and stars [1]. Dynamo has been observed in several numerical simulations and laboratory experiments in which a magnetic field is amplified significantly from a seed magnetic field.

Some of the important problems in dynamo are related to the dynamo transition, e.g., what is the nature of the bifurcation from the fluid state to the dynamo state with a magnetic field?, how is chaos generated in dynamo?, what leads to the reversals of magnetic field?, how is dynamo affected by forcing and geometry?, etc. In this letter we attempt to understand the nature of bifurcations and the routes to chaos in dynamo under the Taylor-Green (TG) forcing.

The governing equations for dynamo are

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + (\mathbf{B} \cdot \nabla) \mathbf{B} + \nu \nabla^2 \mathbf{v} + \mathbf{F}, \quad (1)$$

$$\partial_t \mathbf{B} + (\mathbf{v} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{v} + \eta \nabla^2 \mathbf{B}, \quad (2)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4)$$

where \mathbf{v} is the velocity, \mathbf{B} is the magnetic field, p is the total pressure (thermal+magnetic), \mathbf{F} is the external force field, ν is the kinematic viscosity, and η is the magnetic diffusivity. Some of the important parameters that characterise dynamo are the magnetic Prandtl number $P_M = \nu/\eta$, the Reynolds number $Re = UL/\nu$, and the magnetic Reynolds number $R_M = UL/\eta$, where U and L are the characteristic velocity and length of the system respectively. Only two of the above three parameters are independent ($R_M = Re * P_M$). Note that the range of magnetic Prandtl number P_M observed in nature is extremely large. Liquid

metals and solar plasma typically have small P_M (10^{-5} to 10^{-2}), while interstellar medium has typically large P_M (of the order of 10^{14}). There are other important parameters in dynamo, for example, Rossby number, Ekman number, Rayleigh number, etc. To abstract and simplify the physics of dynamo transition, in the present letter we focus only on basic MHD equations [Eqs. (1-4)] that have only two independent parameters, P_M and Re .

In MHD systems, forcing is typically applied only to the velocity field. A steady magnetic field is generated beyond a certain critical forcing. The magnetic Reynolds number corresponding to this forcing is called the critical magnetic Reynolds number (R_M^c). The first bifurcation is generally a pitchfork bifurcation. Subsequent secondary bifurcations yield more complex configurations like periodic, quasiperiodic, and chaotic magnetic fields. In this letter we will investigate the bifurcations and routes to chaos for $P_M = 1$ under the TG forcing.

Dynamo transition has been studied extensively using experiments, numerical simulations, and theoretical modelling. The Riga dynamo experiment [2], which relies on the concept of the Ponomarenko dynamo [3], was the first successful liquid metal experiment in which dynamo was observed. Dynamo has also been observed in the Karlsruhe experiment [4] in which the flow pattern is helical. Recently Monchaux *et al.* [5] performed dynamo experiments on liquid sodium ($P_M \sim 10^{-5}$) confined within a horizontal cylinder. The fluid was forced using two fans at the two ends of the cylinder. The speeds of the two fans were used as the main controlling parameters. This experiment is called VKS (von Kármán Sodium) due to the nature of its forcing and the velocity configurations. Monchaux *et al.* [5] observed dynamo for $R_M \approx 30$ for equal and opposite forcing frequencies of the two fans as a result of a supercritical pitchfork bifurcation. The resulting magnetic field is axisymmetric and is constant in time. By changing the relative frequencies of the fans, the VKS experimental team could obtain a variety of dynamo solutions including periodic, quasiperiodic, and chaotic magnetic field configurations [6]. They also observed reversals of the magnetic field.

Various researchers have simulated dynamo using direct numerical simulations (DNS) with random, TG, ABC, and Roberts forcing and observed dynamo in both low and high P_M regimes. Variations of R_M^c as a function of P_M have been investigated in these studies. TG forcing has certain similarity with the von Kármán flow configuration and the VKS experiment, hence it has become quite popular. Nore *et al.* [7] demonstrated the existence of dynamo action under TG forcing. Ponty *et al.* [8] applied TG forcing and reported that R_M^c increases sharply with P_M^{-1} as turbulence sets in, and then saturates. Ponty *et al.* [9] observed subcritical dynamo transition by changing R_M in their simulations. Mininni [10] observed various dynamical regimes including time-periodic oscillations and well-defined spatial structures. Dubrulle *et al.* [11] reported various bifurcations in TG flows for both hydrodynamics and MHD simulations.

In another set of simulations, Schekochihin *et al.* [12] and Iskakov *et al.* [13] applied non-helical random forcing and showed the existence of dynamo for both large P_M and small P_M . Podvigina [14], Mininni [15], and Mininni and Montgomery [16] simulated helical dynamo using ABC and Roberts forcing. Podvigina [14] simulated dynamo with ABC forcing and studied various magnetic field states including chaos. The author related these states with inherent symmetries of the system. Gissinger *et al.* [17] generated dynamo by axisymmetric forced flow in a spherical domain; they also observed chaotic magnetic field reversals in the same geometry. Morin and Dormy [18] studied dynamo in a rotating spherical shell and observed either supercritical or subcritical dynamo transition depending on the chosen set of parameter values. Glatzmier and Roberts [19] simulated geodynamo and observed several interesting phenomena including the reversals of the magnetic field.

Dynamo transitions have also been studied using low-dimensional models that are constructed through heuristic arguments, or from partial differential equations using Galerkin projections. These models are amenable to analytical investigations of bifurcations due to smaller number of modes. Rikitake model or disk dynamo, which belongs to the former

category, is a discrete model with conducting discs and current-carrying wires [20]. This model shows complex dynamical behaviour including constant, periodic, quasiperiodic, and chaotic magnetic fields. The model of Petrélis and coworkers [21] is based on amplitude equations and symmetry arguments. They attempted to understand the origin of various dynamo states and magnetic field reversals using the saddle-node bifurcation in their low-dimensional model [21]. Models based on Galerkin projections have been constructed by Donner *et al.* [22] and Verma *et al.* [23]. The magnetic field in the six-mode model of Verma *et al.* [23] is generated by a supercritical pitchfork bifurcation at the transition point. This model exhibits only a constant magnetic field (in time), and it does not show complex dynamical behaviour. On the contrary, the 152-mode model of Donner *et al.* [22] shows various patterns including constant, time-periodic, quasiperiodic, and chaotic states for $P_M = 1$. There are other dynamo models based on scale separations, e.g., α -dynamo, β -dynamo, etc. [1].

In this letter we numerically study the dynamo transition for TG forcing given by

$$\mathbf{F}(k_0) = F_0 \begin{bmatrix} \sin(k_0 x) \cos(k_0 y) \cos(k_0 z) \\ -\cos(k_0 x) \sin(k_0 y) \cos(k_0 z) \\ 0 \end{bmatrix}, \quad (5)$$

where F_0 is the forcing amplitude, and k_0 is the wavenumber of the forcing taken as 2 in this work. Note that the helicity of the force, $(\nabla \times \mathbf{F}) \cdot \mathbf{F} = 0$ everywhere. Nore *et al.* [7] and Ponty *et al.* [8] however argue that local fluctuations in kinetic helicity are generated by the above forcing. We solve Eqs. (1-4) using TARANG [24], a pseudo-spectral code, in a 3D box of dimensions 2π on each side for $P_M = 1$. We apply Runge-Kutta fourth order scheme for time advancement. The time increment dt is determined using the CFL condition ($dt = \Delta x / (20\sqrt{E^u})$, where Δx is the grid size, and E^u is the total kinetic energy). The number of grid points used in our simulation is 64^3 . Our runs are dealiased using 2/3 rule. The range of Reynolds number investigated is from 6 to 160, for which our simulations are well resolved as $k_{max}\eta$ (the largest wavenumber times the Kolmogorov length) is always greater than 1.3. This observation is corroborated from the well resolved kinetic energy spectrum $\langle E^u(k) \rangle$ and magnetic energy spectrum $\langle E^b(k) \rangle$ shown in fig. 1 for $\nu = \eta = 0.1$, and $F_0 = 4.8$. The number of interacting Fourier modes of our dynamo system is 64^3 , which is quite large. Our simulations however reveal that only a small fraction of them carry most of the energy. In many numerical runs with different F_0 's, we observe that the most prominent velocity Fourier modes are $(\pm 2, \pm 2, \pm 2)$, $(\pm 4, \pm 4, \pm 4)$, $(\pm 4, \pm 4, 0)$, $(0, \pm 8, \pm 4)$, and the most prominent magnetic Fourier modes are $(0, 0, \pm 1)$, $(0, 0, \pm 2)$, $(0, 0, \pm 3)$, $(\pm 2, \pm 2, \mp 3)$, $(\mp 2, \mp 2, \pm 1)$. Here the three arguments refer to x , y , and z components of the wavenumber. Note that other modes like $(\pm 4, 0, \pm 4)$ for velocity field are also present due to symmetry. The most energetic velocity Fourier mode is $(\pm 2, \pm 2, \pm 2)$ due to $k_0 = 2$ of the TG forcing. Among the magnetic modes, the most dominant modes are $\mathbf{B}(0, 0, 1)$ and $\mathbf{B}(0, 0, 2)$. The mode $\mathbf{B}(0, 0, 1)$ is generated due to the nonlinear interactions between $(\mathbf{v}(2, 2, 2), \mathbf{B}(-2, -2, -1))$, and the mode $\mathbf{B}(0, 0, 2)$ is generated by $(\mathbf{v}(2, 2, 2), \mathbf{B}(-2, -2, 0))$. We observe a dynamic interplay between $\mathbf{B}(0, 0, 1)$ and $\mathbf{B}(0, 0, 2)$ modes. The other important magnetic mode is $\mathbf{B}(0, 0, 3)$ which participates with $(\mathbf{v}(2, 2, 2), \mathbf{B}(-2, -2, -1))$.

We perform DNS for various forcing parameter F_0 . Initial transients are discarded and only the steady-state configurations are analysed. For $F_0 = 4.8$, the steady-state velocity and magnetic fields (snapshots of the magnitudes) are shown in figs. 2(a,b) respectively. These figures indicate that the TG forcing yields well defined velocity and magnetic structures. For $k_0 = 2$, chosen for our runs, the simulation box has 16 TG cells for the velocity field. As shown in fig. 2(b), the magnetic energy is concentrated in two major slabs along with two minor slabs. We also observe that $|B_z| \ll |B_x|, |B_y|$. The above configuration is due to the prominence of the modes $\mathbf{B}(0, 0, 1)$ and $\mathbf{B}(0, 0, 2)$. Note that the z components for these modes are zero due to the incompressibility condition. Thus, we can understand the global TG structures in terms of the dominant Fourier modes. Figure 3 contains the vector plot

of the magnetic field (x and y components only) for a cross-section of fig. 2(b) at $z = 4.08$. The magnetic field in this plane is approximately along -45° . These field configurations are similar to those reported by Mininni *et al.* [10].

To explore various bifurcations of the dynamo state, we vary F_0 in the interval $[1 : 40]$. We observe pure fluid solutions ($E^b = 0$) till $F_0 = 3.9$, after which nonzero steady-state magnetic field ($E^b > 0$) emerges with each component of the magnetic field as constant (in time). The magnetic Reynolds number at this transition regime is approximately 19.9, hence $R_M^c \simeq 19.9$. The number of variables of our dynamo system is unfortunately rather large ($\sim 64^3$). We could however focus on the most energetic modes that determine the system dynamics. We investigate the time series of these modes, and obtain various dynamo states: constant (fixed point), periodic, quasiperiodic, and chaotic magnetic fields. In fig. 4 we show the time series of a fixed point ($F_0 = 3.9$), a periodic state ($F_0 = 10$), a quasiperiodic state ($F_0 = 36$), and a chaotic state ($F_0 = 4.8$) of the dynamo.

Different states of a dynamical system are elegantly illustrated in the bifurcation diagram that contains information about the birth of new states. In fig. 5 we construct a bifurcation diagram by plotting the time averaged value of the amplitude of the magnetic mode $\langle B(0, 0, 1) \rangle$ for different F_0 (obtained from around 60 DNS runs). Nonzero magnetic field appears at around $F_0 = 3.9$ through a pitchfork bifurcation. There is no hysteresis near the onset of dynamo as F_0 is increased or decreased. Hence, the dynamo transition is through a supercritical pitchfork bifurcation. After the primary instability (or bifurcation), we observe different kinds of dynamo states like constant, periodic, quasiperiodic, and chaotic magnetic fields as evident from the bifurcation diagram. The windows of these states appear for various range of F_0 , e.g., chaos appears for $F_0 = 11 - 12$. We also find windows of F_0 where $\langle B(0, 0, 1) \rangle$ becomes relatively small, but $\langle B(0, 0, 2) \rangle$ becomes significant. The dynamics of interchange of energy between $\mathbf{B}(0, 0, 1)$ and $\mathbf{B}(0, 0, 2)$ is not apparent at present, and it will be studied in future.

A careful analysis of the dynamics for a given forcing reveals coexistence of multiple dynamo states. We illustrate this feature using state space or phase space. The phase space of our dynamo is very large ($\sim 64^3$). However, projections of the phase space on the subspace formed by the most energetic modes contain most of the information of the system. For $F_0 = 16$, two different sets of initial conditions yield either a fixed point or a periodic solution. These two states are illustrated in figs. 6(a,b) using projections of the phase space on $(B(0, 0, 1)-B(0, 0, 3))$ plane. At $F_0 = 36$, the two different coexisting dynamo states involve quasiperiodic and chaotic magnetic fields as illustrated in figs. 6(c,d).

The bifurcation diagram (fig. 5) is quite complex, and it is not possible to probe all the secondary bifurcations using DNS. In the present letter we focus on a narrow window ($F_0 = 4.6 - 4.8$) of fig. 5 to investigate how chaos appears in this window. As we will describe below, here we observe a period-doubling route to chaos. To understand the transition to chaos better, we plot the phase space projections on the $(B(0, 0, 1)-B(0, 0, 3))$ plane. At around $F_0 = 4.6$ we observe a fixed point or a constant magnetic field as evident from fig. 7(a). At $F_0 \simeq 4.73$, the fixed point bifurcates to a periodic solution (limit cycle) through a Hopf bifurcation (also shown in fig. 7(a)). The power spectral density (PSD) plot of the time series for this F_0 exhibits a peak at a single frequency $f_1 = f \simeq 0.018$ (in non-dimensional units) as shown in the right side of fig. 7(a). As we increase F_0 , we observe period-2, period-4, and period-8 solutions at $F_0 = 4.77, 4.777$, and 4.778 respectively. These new states are generated through the “period-doubling bifurcations”. Projections of the phase space on the $(B(0, 0, 1)-B(0, 0, 3))$ plane along with the power spectra for these three states are shown in figs. 7(b,c,d) respectively. The power spectra clearly indicate peaks at subharmonics $f_2 = f/2$, $f_3 = f/4$, $f_4 = f/8$ corresponding to the period-2, period-4, and period-8 dynamo states. At $F_0 \simeq 4.8$, the system becomes chaotic as evident from the phase space projection and the broad frequency spectrum shown in fig. 7(e). The above observations indicate that chaos appears at $F_0 \simeq 4.8$ through the period-doubling route to chaos. The value of magnetic Reynolds number for these states is around 22. The bifurcation

diagram (fig. 5) exhibits several other windows of chaos whose origin has not been explored in this letter.

In conclusion, our numerical simulations of dynamo for $P_M = 1$ with Taylor-Green forcing reveal that the dynamo transition takes place through a supercritical pitchfork bifurcation. After the primary bifurcation, the system exhibits several windows of constant, periodic, quasiperiodic and chaotic solutions. We also find multiple coexisting attractors for a given parameter; different initial conditions take the system to one or the other attractor. A careful analysis of one of the chaotic windows reveals that the dynamo becomes chaotic through a period-doubling route to chaos.

Our numerical simulations reveal several dynamo states that have been observed in experiments (such as VKS), earlier numerical simulations, and low-dimensional models. The geometry and the forcing of our simulations however are simpler than those of experiments. Yet the above similarities may be due to certain inherent common features of dynamo. Future numerical simulations with more realistic geometry and forcing functions will reveal valuable insights into this puzzle.

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Figure captions

Fig. 1: The kinetic energy spectrum $\langle E^u(k) \rangle$ and the magnetic energy spectrum $\langle E^b(k) \rangle$ for $F_0 = 4.8$ for which the magnetic field is chaotic. The spectra indicates that the simulations are well resolved.

Fig. 2: Flow structure of the Taylor-Green (TG) flow: (a) volume rendering of the velocity field magnitude that illustrates 16 TG cells due to $k_0 = 2$; (b) volume rendering of the magnetic field magnitude illustrating two major and two minor slabs. This emphasizes the dominance of $\mathbf{B}(0, 0, 1)$ and $\mathbf{B}(0, 0, 2)$ magnetic modes.

Fig. 3: The magnetic field vector plot for a cross-section of fig. 2(b) at $z = 4.08$.

Fig. 4: The time series of the real part of $B_x(0, 0, 1)$ for $F_0 = 3.9$ (a), 10(b), 36(c) and 4.8(d).

Fig. 5: Bifurcation diagram exhibiting various dynamo states of our simulations with FP = fixed point, P = periodic state, QP = quasiperiodic state, and C = chaotic state. The time-averaged amplitude of $B(0, 0, 1)$ is plotted for various F_0 . Multiple attractors are observed at $F_0 = 16$ and $F_0 = 36$ (see fig. 6). The circled region of the inset exhibits period-doubling route to chaos (see fig. 7).

Fig. 6: Phase space projections on the $(B(0, 0, 1)-B(0, 0, 3))$ plane for $F_0 = 16$ (a,b) and $F_0 = 36$ (c,d) exhibiting multiple attractors. Figures (a,b) show coexisting fixed point and periodic states for two different initial conditions. Figures (c,d) show coexisting quasiperiodic and chaotic states.

Fig. 7: Phase space projections on the $(B(0, 0, 1)-B(0, 0, 3))$ plane and the power spectra exhibiting period-doubling route to chaos: (a) a fixed point at $F_0 = 4.6$, and period-1 solution at $F_0 = 4.73$; the power spectrum in RHS exhibits a peak at $f \simeq 0.018$ (non-dimensional units) for $F_0 = 4.73$; (b) period-2 solution at $F_0 = 4.77$ with power spectrum showing two peaks at f_1 and $f_2(= f/2)$; (c) period-4 solution at $F_0 = 4.777$ with power spectrum with three peaks at f_1, f_2 and $f_3(= f/4)$; (d) period-8 solution at $F_0 = 4.778$ with power spectrum showing four peaks at f_1, f_2, f_3 and $f_4(= f/8)$; (e) chaotic solution with a broad-band power spectrum.

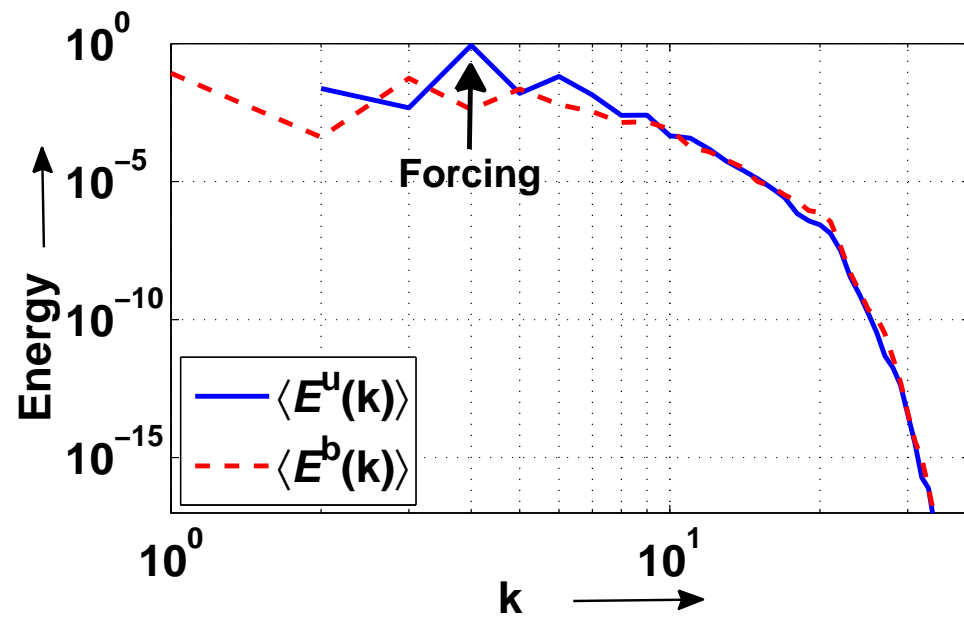


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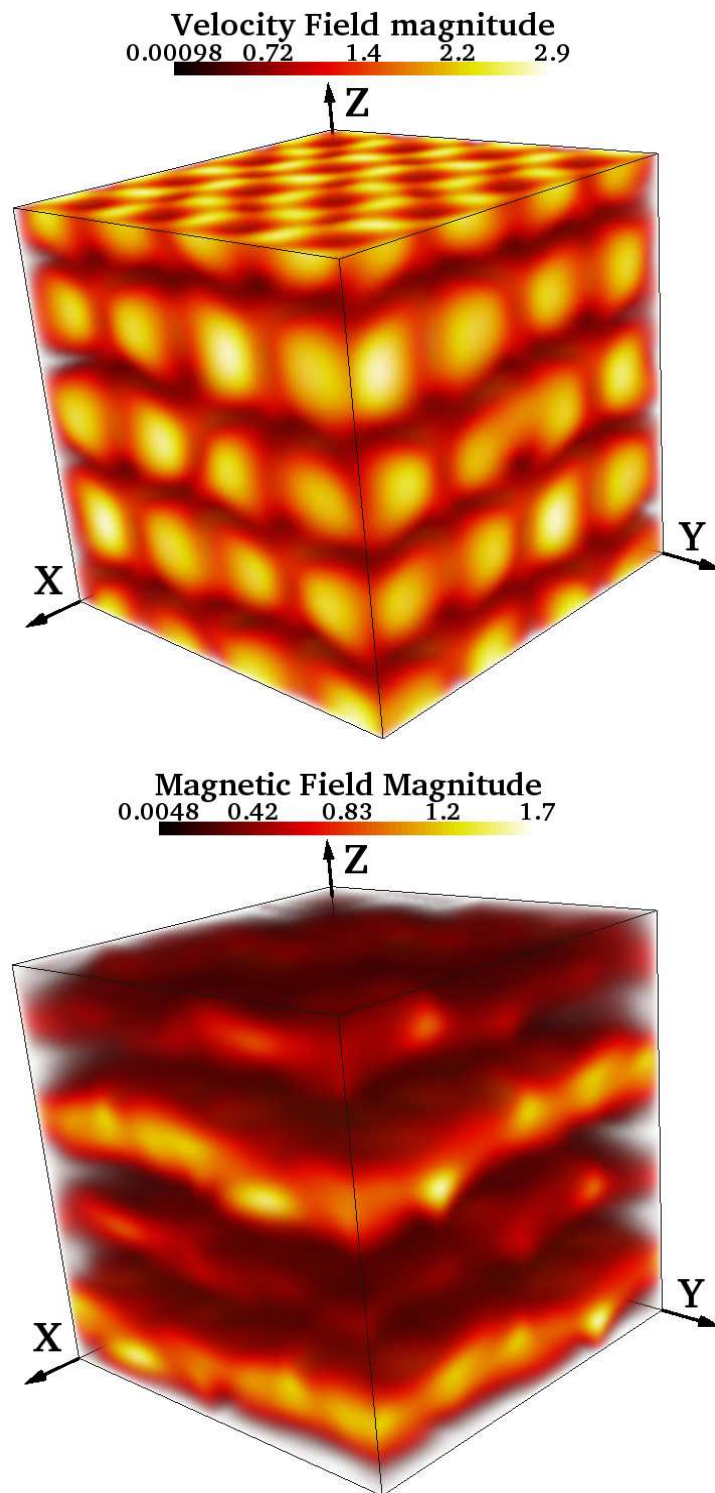


Fig. 2:

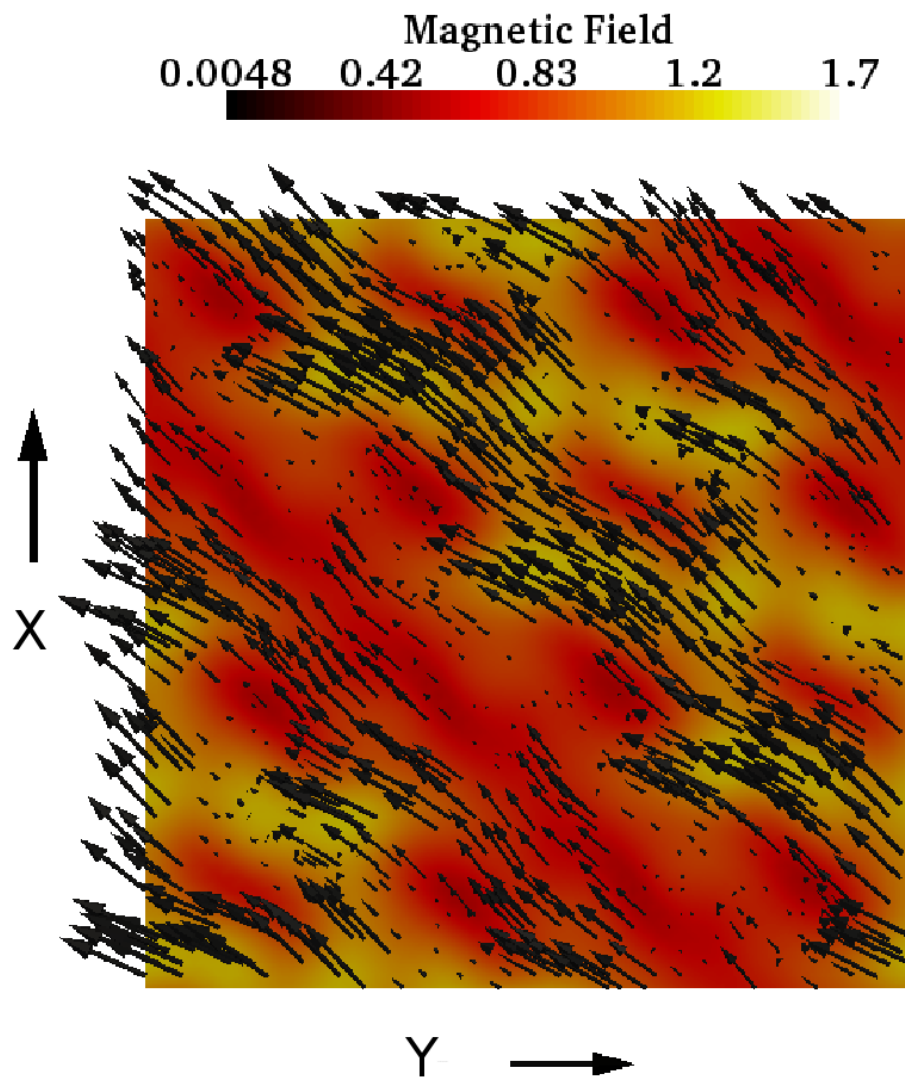


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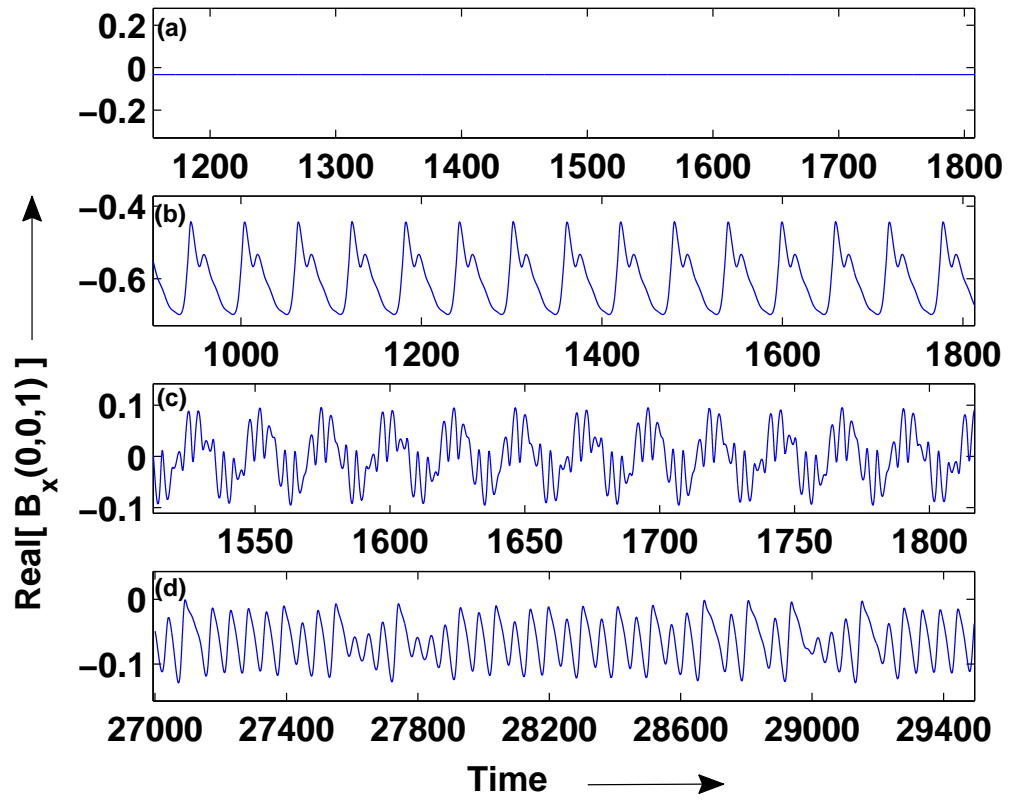


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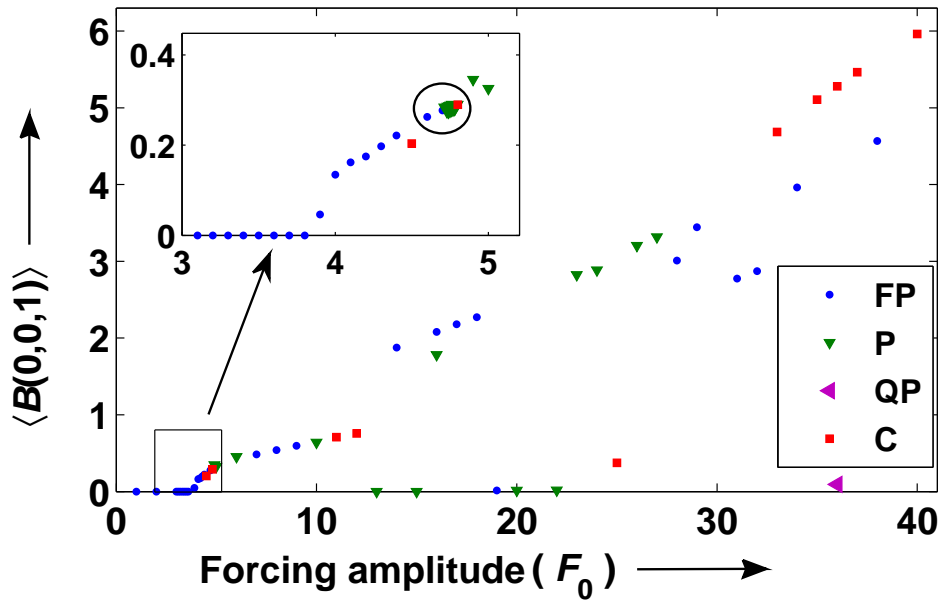


Fig. 5:

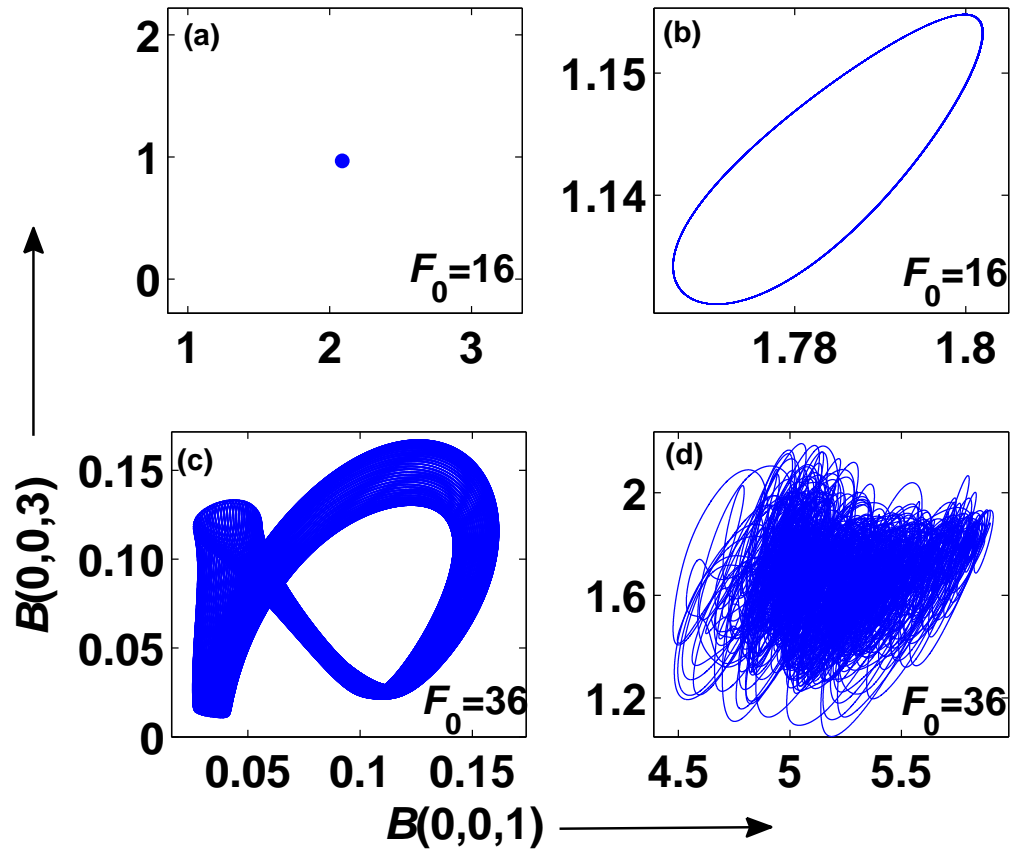


Fig. 6:

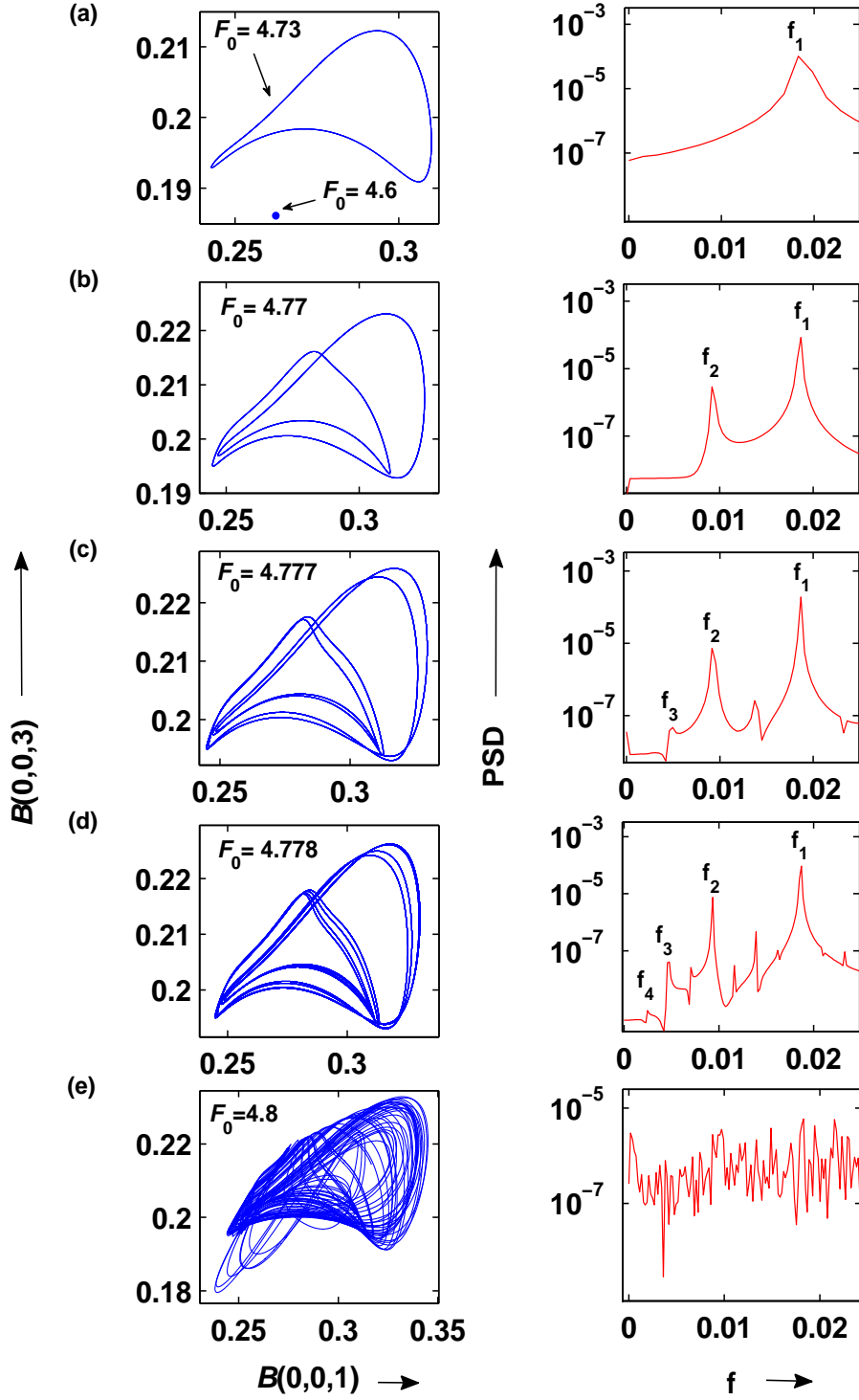


Fig. 7: